

Labelled OSPA metric for fixed and known number of targets

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Abstract—The evaluation of multiple target tracking algorithms with labelled sets can be done using the labelled optimal subpattern assignment (LOSPA) metric. In this paper, we provide the expression of the same metric for fixed and known number of targets when vector notation is used.

Index Terms—Target labelling, multiple target tracking, random finite sets

I. INTRODUCTION

Multitarget tracking systems should solve two basic problems. The first one is to estimate the number of targets and their states at the current time. The second one is to connect target state estimates that belong to the same target along time to form tracks. The conventional way of building tracks in the random finite set framework (RFS) [1] is to attach a label to the individual target states [2], [3].

Labels have two important properties: they are unique (no two targets can have the same label) and they are fixed over time. Labels were used for track formation in [4], [5] using a vector-based formulation and in [2], [3] using the RFS framework. The approaches of [4], [5] and [2], [3] are equivalent due to the bijection between the labelled RFS state and the hybrid labelled multitarget state vector [6, Appendix B]. For the same reason, for fixed and known number of targets, representing the multitarget state as a vector is equivalent to a labelled set. One way to evaluate performance of tracking algorithms based on labelled set is using the labelled optimal subpattern assignment (LOSPA) metric [7].

In some cases, it is convenient to assume that the number of targets is fixed and known [8], [9]. This way we can study some properties of tracking algorithms more easily. In these cases, it is usually useful to use vector notation, in which labels are implicit in the ordering of the components of a vector, to denote a labelled set. The problem is that the LOSPA metric in [7] is defined with explicit labels. In this paper, we fill this gap and provide an expression for this metric when the number of targets is fixed and known and vector notation is used.

This paper is organised as follows. In Section II, we introduce the two equivalent representations of the multitarget state based on a labelled set and a vector. We provide the

expression for the LOSPA metric using vector notation in Section III.

II. LABELLED SET AND VECTOR NOTATION

In this paper, we make the following assumption

- A The number of targets is fixed and known

Under Assumption A, the labelled set that contains the targets and the labels is represented as $\left\{ \left[(\mathbf{x}_1^k)^T, l_1 \right]^T, \left[(\mathbf{x}_2^k)^T, l_2 \right]^T, \dots, \left[(\mathbf{x}_t^k)^T, l_t \right]^T \right\}$ where $l_j \in \mathbb{R}$ represents the j th label, $\mathbf{x}_j^k \in \mathbb{R}^{n_x}$ is the state vector at time k for target with label l_j and T denotes transpose. Labels are unique, assigned deterministically and do not change with time. Therefore, the same information of the labelled set $\left\{ \left[(\mathbf{x}_1^k)^T, l_1 \right]^T, \left[(\mathbf{x}_2^k)^T, l_2 \right]^T, \dots, \left[(\mathbf{x}_t^k)^T, l_t \right]^T \right\}$ is contained in the multitarget state vector $\mathbf{X}^k = \left[(\mathbf{x}_1^k)^T, (\mathbf{x}_2^k)^T, \dots, (\mathbf{x}_t^k)^T \right]^T \in \mathbb{R}^{tn_x}$. The labels of the labelled set are implicit in the ordering inherent in the multitarget state vector components and we can establish a bijection between the multitarget state vector and the labelled set.

Under Assumption A, it is usually more convenient to use the multitarget state vector than the labelled set because we do not have to carry along the explicit labels. This is for example useful when performing Bayesian inference.

III. LABELLED OSPA METRIC WITH VECTOR NOTATION

Here, we provide the expression of the LOSPA metric for labelled sets defined in [7] when we use the vector notation under Assumption A. We also prove it is a metric with this notation for completeness. It should be noted that the triangle inequality for LOSPA metric using labelled sets is proved in [10].

We represent the permutations of vector $[1, \dots, t]^T$ as vectors $\phi_i = [\phi_{i,1}, \dots, \phi_{i,t}]^T$ $i \in \{1, \dots, t!\}$. Then, the labelled OSPA (LOSPA) distance between multitarget vectors $\mathbf{A}^k = \left[(\mathbf{a}_1^k)^T, (\mathbf{a}_2^k)^T, \dots, (\mathbf{a}_t^k)^T \right]^T \in \mathbb{R}^{tn_x}$ and $\mathbf{B}^k = \left[(\mathbf{b}_1^k)^T, (\mathbf{b}_2^k)^T, \dots, (\mathbf{b}_t^k)^T \right]^T \in \mathbb{R}^{tn_x}$ is

$$d(\mathbf{A}^k, \mathbf{B}^k) = \left(\frac{1}{t} \min_{i \in \{1, \dots, t!\}} \left[\sum_{j=1}^t b^p \left(\mathbf{a}_j^k, \mathbf{b}_{\phi_{i,j}}^k \right) + \alpha^p \bar{\delta} [j - \phi_{i,j}] \right] \right)^{1/p} \quad (1)$$

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Table I
LOSPA BETWEEN $\hat{\mathbf{X}}^k$ AND $\mathbf{X}^k = [-10, 0, 10]^T$

Estimate $\hat{\mathbf{X}}^k$	LOSPA ($\alpha = 0.1$)	LOSPA ($\alpha = 1$)
$[-10.1, 0.1, 10.1]^T$	0.1	0.1
$[0.1, -10.1, 10.1]^T$	$\sqrt{0.1^2 + 0.02/3}$	$\sqrt{0.1^2 + 2/3}$
$[10.1, -10.1, 0.1]^T$	$\sqrt{0.1^2 + 0.03/3}$	$\sqrt{0.1^2 + 3/3}$

where $\bar{\delta}[\cdot]$ is the complement of the Kronecker delta, i.e., $\bar{\delta}[j] = 0$ if $j = 0$ and $\bar{\delta}[j] = 1$ otherwise, $\alpha > 0$, $1 \leq p < \infty$ and $b(\cdot, \cdot)$ is a metric on the space \mathbb{R}^{n_x} . In [7] the authors include another parameter p' , we set $p' = p$ for simplicity. Function $d(\cdot, \cdot)$ is a metric as it satisfies the axioms of identity, symmetry and triangle inequality. The identity and symmetry are straightforward to check. The proof of the triangle inequality is given in Appendix A. It should be noted that if $\alpha = 0$, we get the optimal subpattern assignment metric (OSPA) without cut-off distance [11] and not the LOSPA. In Appendix C, we prove that this metric is equivalent in the labelled set domain.

Illustrative example: We illustrate how the LOSPA metric works in a simple example. Let us assume there are three unidimensional targets and the multitarget state is $\mathbf{X}^k = [-10, 0, 10]^T$. That is, target 1 is at -10, target 2 is at 0 and target 3 is at 10. We use the Euclidean metric for $b(\cdot, \cdot)$ with $p = 2$. The LOSPA between \mathbf{X}^k and several estimates $\hat{\mathbf{X}}^k$, which only differ in their labelling, are given in Table I. As all the estimates only differ in their labelling, they have the same OSPA, which is 0.1. This implies that all the estimates have the same accuracy as regards where the targets are. However, the first estimate is closer in the LOSPA sense than the rest. The higher α is, the more the metric penalises wrong labelling/ordering.

APPENDIX A

In this appendix we prove the triangle inequality of the LOSPA metric, which is given by (1). We want to show that

$$d(\mathbf{X}^k, \mathbf{Y}^k) \leq d(\mathbf{X}^k, \mathbf{Z}^k) + d(\mathbf{Z}^k, \mathbf{Y}^k) \quad (2)$$

As (9) in Appendix B is met for any $m \in \{1, \dots, t\}$ and $i \in \{1, \dots, t\}$, we can write

$$\begin{aligned} & \min_{i \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{i,j}]} \\ & \leq \min_{m \in \{1, \dots, t\}} \min_{i \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{m,j}]} \\ & \quad + \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[\phi_{m,j} - \phi_{i,j}]} \\ & = \min_{m \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{m,j}]} \end{aligned}$$

$$+ \min_{i \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[\phi_{m,j} - \phi_{i,j}]}$$

By using a change of variables $\phi_{m,j} = j$ for $j = 1 \dots t$, we get

$$\begin{aligned} & \min_{i \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[\phi_{m,j} - \phi_{i,j}]} \\ & = \min_{i \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{z}_j^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{i,j}]} \end{aligned}$$

which is independent of m . Therefore,

$$\begin{aligned} & \min_{i \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{i,j}]} \\ & \leq \min_{m \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{m,j}]} \\ & \quad + \min_{i \in \{1, \dots, t\}} \sqrt[p]{\sum_{j=1}^t b^p(\mathbf{z}_j^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{i,j}]} \end{aligned}$$

Then, we can write

$$\begin{aligned} & \left(\frac{1}{t} \min_{i \in \{1, \dots, t\}} \left[\sum_{j=1}^t b^p(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{i,j}] \right] \right)^{1/p} \\ & \leq \left(\frac{1}{t} \min_{i \in \{1, \dots, t\}} \left[\sum_{j=1}^t b^p(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{m,j}] \right] \right)^{1/p} \\ & \quad + \left(\frac{1}{t} \min_{i \in \{1, \dots, t\}} \left[\sum_{j=1}^t b^p(\mathbf{z}_j^k, \mathbf{y}_{\phi_{i,j}}^k) + \alpha^p \bar{\delta}[j - \phi_{i,j}] \right] \right)^{1/p} \end{aligned}$$

Using (1), we complete the proof of the triangle inequality.

APPENDIX B

This appendix provides a subsidiary result that is necessary for the proof of the triangle inequality in Appendix A. Using the fact that $b(\cdot, \cdot)$ is a metric

$$\begin{aligned} b(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k) & \leq b(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k) + b(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k) \\ b^p(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k) & \leq \left(b(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k) + b(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k) \right)^p \end{aligned} \quad (3)$$

In addition

$$\begin{aligned} \alpha \bar{\delta}[j - \phi_{i,j}] & \leq \alpha \bar{\delta}[j - \phi_{m,j}] + \alpha \bar{\delta}[\phi_{m,j} - \phi_{i,j}] \\ (\alpha \bar{\delta}[j - \phi_{i,j}])^p & \leq (\alpha \bar{\delta}[j - \phi_{m,j}] + \alpha \bar{\delta}[\phi_{m,j} - \phi_{i,j}])^p \end{aligned} \quad (4)$$

Using (3) and (4), we get

$$\begin{aligned} & b^p(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k) + (\alpha \bar{\delta}[j - \phi_{i,j}])^p \\ & \leq \left(b(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k) + b(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k) \right)^p \\ & \quad + (\alpha \bar{\delta}[j - \phi_{m,j}] + \alpha \bar{\delta}[\phi_{m,j} - \phi_{i,j}])^p \end{aligned} \quad (5)$$

$$\begin{aligned}
& \sum_{j=1}^t b^p \left(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k \right) + (\alpha \bar{\delta} [j - \phi_{i,j}])^p \\
& \leq \sum_{j=1}^t \left(b \left(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k \right) + b \left(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k \right) \right)^p \\
& \quad + (\alpha \bar{\delta} [j - \phi_{m,j}] + \alpha \bar{\delta} [\phi_{m,j} - \phi_{i,j}])^p \quad (6)
\end{aligned}$$

$$\begin{aligned}
& \left[\sum_{j=1}^t b^p \left(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k \right) + (\alpha \bar{\delta} [j - \phi_{i,j}])^p \right]^{1/p} \\
& \leq \left[\sum_{j=1}^t \left(b \left(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k \right) + b \left(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k \right) \right)^p \right. \\
& \quad \left. + (\alpha \bar{\delta} [j - \phi_{m,j}] + \alpha \bar{\delta} [\phi_{m,j} - \phi_{i,j}])^p \right]^{1/p} \quad (7)
\end{aligned}$$

Using Minkowski inequality [12]

$$\begin{aligned}
& \left[\sum_{j=1}^t \left(b \left(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k \right) + b \left(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k \right) \right)^p \right. \\
& \quad \left. + (\alpha \bar{\delta} [j - \phi_{m,j}] + \alpha \bar{\delta} [\phi_{m,j} - \phi_{i,j}])^p \right]^{1/p} \\
& \leq \left[\sum_{j=1}^t b^p \left(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k \right) + \alpha^p \bar{\delta} [j - \phi_{m,j}] \right]^{1/p} \\
& \quad + \left[\sum_{j=1}^t b^p \left(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k \right) + \alpha^p \bar{\delta} [\phi_{m,j} - \phi_{i,j}] \right]^{1/p} \quad (8)
\end{aligned}$$

Using (8) into (7), we get

$$\begin{aligned}
& \left[\sum_{j=1}^t b^p \left(\mathbf{x}_j^k, \mathbf{y}_{\phi_{i,j}}^k \right) + \alpha^p \bar{\delta} [j - \phi_{i,j}] \right]^{1/p} \\
& \leq \left[\sum_{j=1}^t b^p \left(\mathbf{x}_j^k, \mathbf{z}_{\phi_{m,j}}^k \right) + \alpha^p \bar{\delta} [j - \phi_{m,j}] \right]^{1/p} \\
& \quad + \left[\sum_{j=1}^t b^p \left(\mathbf{z}_{\phi_{m,j}}^k, \mathbf{y}_{\phi_{i,j}}^k \right) + \alpha^p \bar{\delta} [\phi_{m,j} - \phi_{i,j}] \right]^{1/p} \quad (9)
\end{aligned}$$

for any $m \in \{1, \dots, t\}$.

APPENDIX C

In this appendix we show that the metric used in this paper is equivalent in the set domain under Assumption A. The labelled OSPA metric $d_s(\cdot, \cdot)$ in the set domain [7] requires the definition of the labelled sets

$$\begin{aligned}
A^k &= \left\{ \left[(\mathbf{a}_1^k)^T, l_1 \right]^T, \left[(\mathbf{a}_2^k)^T, l_2 \right]^T, \dots, \left[(\mathbf{a}_t^k)^T, l_t \right]^T \right\} \\
B^k &= \left\{ \left[(\mathbf{b}_1^k)^T, l_1 \right]^T, \left[(\mathbf{b}_2^k)^T, l_2 \right]^T, \dots, \left[(\mathbf{b}_t^k)^T, l_t \right]^T \right\}
\end{aligned}$$

where l_1, \dots, l_t are the explicit labels of the targets that must be used in the set approach. Then,

$$d_s(A^k, B^k)$$

$$\begin{aligned}
& = \left(\frac{1}{t} \min_{i \in \{1, \dots, t\}} \left[\sum_{j=1}^t b^p \left(\mathbf{a}_j^k, \mathbf{b}_{\phi_{i,j}}^k \right) + \alpha^p \bar{\delta} [l_j - l_{\phi_{i,j}}] \right] \right)^{1/p} \\
& = \left(\frac{1}{t} \min_{i \in \{1, \dots, t\}} \left[\sum_{j=1}^t b^p \left(\mathbf{a}_j^k, \mathbf{b}_{\phi_{i,j}}^k \right) + \alpha^p \bar{\delta} [j - \phi_{i,j}] \right] \right)^{1/p} \\
& = d(A^k, B^k)
\end{aligned}$$

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